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LETTER TO THE EDITOR

A geometric proof of the Kochen–Specker no-go theorem

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Abstract. We give a short geometric proof of the Kochen–Specker no-go theorem for non-contextual hidden variables models.

1. Introduction

The fundamental theorem of Kochen and Specker (1967) shows that any hidden-variable theory for quantum measurement (on an at least three-dimensional system) must be *contextual*: i.e. in a deterministic theory randomness is explained not just by hidden states in the quantum system under study but also from hidden states in the measurement devices.

The theorem is usually proved by exhibiting a finite collection of vectors in $\mathbb{C}^3$ (actually, $\mathbb{R}^3$ turns out to be enough) such that it is impossible to colour each vector either red or green subject to the following constraints: (i) within any orthogonal triple, exactly one vector is red and the other two are green; (ii) if one vector lies in a (complex) linear combination of another two and those two are both coloured green, then it is coloured green as well. The two constraints are connected to the so-called sum-rule and product-rule associating values of commuting observables. For the preparatory arguments showing why such a construction does supply a proof of the no-go theorem for non-contextual hidden variables models see Peres (1993) or Gill (1995a,b).

The Kochen–Specker proof is based on a construction involving 117 vectors. Actually, the heart of the construction is a special configuration of just ten vectors, which is then chained in three groups of five (with three of the vectors being used twice). Ignored by most authors is an earlier construction by Bell (1966) again based on a special configuration of 13 vectors repeated a number of times. Recently Peres (1991) gave a construction involving just 33 vectors. In his more recent book (Peres 1993) he also shows a construction of Conway and Kochen involving just 31 vectors. This is the world record so far. Peres (1993) and Gill (1995b) also discuss further examples due to Peres, Mermin, and others, involving still fewer vectors, but requiring a higher-dimensional space. A recent contribution of this kind has been made by Cabello *et al* (1996). Such examples do *illustrate* the Kochen–Specker theorem but they do not *prove* it.

Here we present a new construction similar in flavour to the Bell and Kochen–Specker constructions, being based on a repetition of a basic configuration. However, whereas those

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constructions relied on some *analytic* computations to prove their existence, our construction relies on a *geometric* picture—in fact, exactly the same geometric idea used by Cooke *et al* (1985) at the heart of their elementary proof of Gleason's theorem. The recent Peres (1991) and Conway–Kochen (see Peres 1993) constructions have a geometrical aspect but are more *combinatoric* nature. It is therefore largely a matter of mathematical taste which proof is to be preferred. However, we feel there is some virtue in laying a connection with Gleason's theorem (which was also the inspiration of Bell's contribution), and in having a proof which can be 'seen' from a picture without any calculation or lengthy enumeration being necessary. Another (more complicated) geometric proof is given by Galindo (1976), while a more verbal proof using similar ideas to ours is given in the unpublished paper Dorling (1992).

Some authors, e.g. van Fraassen (1991), use Gleason's theorem applied to the continuum of all vectors simultaneously to (allegedly) prove the theorem. In our opinion this cannot be built into a correct proof of the no-go result; see Gill (1995b) for an analysis of what can go wrong. Other authors misinterpret Bell's argument to require continuously many vectors and hence be disqualified but this does not do justice to Bell's argument which in our opinion is both concise and correct.

'How many vectors' are needed in a given argument seems to us a relatively minor point. The theorem is already proved by Bell, Kochen and Specker, and us, after the initial configuration has been shown to exist. Moreover there are different ways of counting vectors (for instance, one might not accept the product-rule but only use the sum rule, and thereby need more vectors). We see no reason not to use anything at our disposal.

2. A geometric lemma

Consider the one-dimensional subspaces corresponding to non-zero, real, linear combinations of three orthogonal vectors in $\mathbb{C}^k$, $k \geqslant 3$. These subspaces may be represented by points on (the surface of) the Northern Hemisphere of the Globe. The original triple is represented by the North Pole together with two points on the Equator whose longitudes differ by 90°.

Now fix a point ψ in the Northern Hemisphere, not at the North Pole nor on the Equator. Consider the great circle through this point which crosses the Equator at the two points differing in longitude by $\pm 90^\circ$ from ψ. Choose one of these equatorial points and call it ψ^{E}. Call the point on the Northern Hemisphere orthogonal to the great circle $\psi^{\perp}$. Its longitude is that of ψ plus 180° and its latitude is 90° minus that of ψ. The triple ψ, ψ^{E}, $\psi^{\perp}$ are orthogonal.

The great circle we just defined has ψ as its most northerly point. We call it *the great circle descent from* ψ.

Starting from a point $\psi = \psi_0$ go down its descent circle some way to a new point ψ_1. Now consider the new great circle descent from ψ_1. Go down some way to a new point ψ_2, and so on. After n steps arrive at ψ_n. Obviously ψ_n is more southerly than ψ_0. Cooke *et al*'s geometric lemma states that one can reach *any* more southerly point than ψ_0 by a finite sequence of great circle descents. For instance, one can fly from Amsterdam to Tokyo by a finite sequence of great circle descents.

The lemma is proved by projecting the Northern Hemisphere *from* the centre of the earth *onto* the horizontal plane tangent to the earth at the North Pole. Lines of constant latitude project onto concentric circles, a great circle descent projects onto a straight line tangent to the circle of constant latitude at its summit.

3. Proof of the theorem

Start with an orthogonal triple. Colour one point red and the other two green. Let the red point be the North Pole and the other two green points be on the Equator. Any further points selected on the Equator get coloured green by the product rule. Take a point ψ at latitute 60° above the Equator. Together with $\psi^\perp$ and ψ^{E} we have a new orthogonal triple. Since ψ^{E} gets coloured green, if ψ is coloured green then $\psi^\perp$ is coloured red. Note that $\psi^\perp$ lies at 30° above the Equator, more southerly than ψ.

Suppose ψ is coloured green. Since any point on its great circle descent is a linear combination of ψ and ψ^{E}, it is also coloured green. Repeating this argument, any point which can be reached by a finite number of great circle descents from ψ is also coloured green. But this applies to $\psi^\perp$, a contradiction.

Therefore ψ is coloured red just like the North Pole. So we have shown that any point within 30° of a red point is also coloured red. Now go in three steps of 30° from the North Pole down to the Equator, then in three steps of 30° along the Equator, then in three steps of 30° back up to the North Pole. One of the three 'corners' of this circuit has to be coloured red, hence they all are, a contradiction. ☐

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